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1994 J. Phys. A: Math. Gen. 27 3895

(http://iopscience.iop.org/0305-4470/27/11/036)

J. Phys. A: Math. Gen. 27 (1994) 3895–3913. Printed in the UK

The two-singular-manifold method: I. Modified Korteweg–de Vries and sine–Gordon equations

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Received 6 January 1994

Abstract. We address the problem of finding, by Painlevé analysis only, the Bäcklund transformation of partial-differential equations (PDEs) having two families of movable singularities with opposite principal parts, such as the modified Korteweg–de Vries (MKdV), sine–Gordon or nonlinear Schrödinger equations. This first paper gives an almost algorithmic method which extends the singular-manifold method of Weiss that is unable to handle these equations. First, with only one singular manifold at a time, we obtain the Darboux transformation. Second, we assume that the ratio of two functions defining the singular manifolds satisfies the most general projective Riccati system with undetermined coefficients; the Darboux transformation then generates a very small number of determining equations, admitting a unique solution, equivalent to the Lax pair of the Zakharov–Shabat–Ablowitz–Kaup–Newell–Segur scheme by the canonical linearization of the Riccati system. The method is here applied to the MKdV and sine–Gordon equations.

1. Introduction

For partial-differential equations (PDEs), one of the most widely accepted definitions of integrability is the existence of a Bäcklund transformation (BT) (Bäcklund 1883, Rogers and Shadwick 1982). A BT between two given PDEs

$$E_1(u,x,t)=0 \qquad E_2(U,X,T)=0 \tag{1}$$

is, by definition (Darboux 1894 vol III ch XII, Matveev and Salle 1991), a pair of relations

$$F_j(u,x,t,U,X,T)=0 \qquad j=1,2 \tag{2}$$

where F_j depends on the derivatives of $u(x,t)$ and $U(X,T)$ such that the elimination of u (respectively U) between (F_1,F_2) implies $E_2(U,X,T)=0$ (respectively $E_1(u,x,t)=0$). When the two PDEs are identical, the BT is called the auto-BT. The auto-BTs for the two PDEs mainly considered in this paper are as follows.

(i) The sine–Gordon (SG) equation. Given two solutions u and U of the SG equation

$$E_1\equiv u_{xt}-\sin u=0 \tag{3}$$

$$E_2\equiv U_{xt}-\sin U=0 \tag{4}$$

the auto-BT is defined as (Lamb 1967)

$$F_1 \equiv (u+U)_x + 4\lambda \sin\frac{u-U}{2} = 0 \qquad F_2 \equiv (u-U)_t + \frac{1}{\lambda}\sin\frac{u+U}{2} = 0 \tag{5}$$

where λ is an arbitrary constant, as shown by the elimination of U (respectively u) between them

$$F_{1,t} + F_{2,x} - \frac{1}{2\lambda}F_1 \cos\frac{u+U}{2} - 2\lambda F_2 \cos\frac{u-U}{2} \equiv 2E_1 \tag{6}$$

$$F_{1,t} - F_{2,x} + \frac{1}{2\lambda}F_1 \cos\frac{u+U}{2} - 2\lambda F_2 \cos\frac{u-U}{2} \equiv 2E_2. \tag{7}$$

Lamb (1967) showed how to obtain from these relations an infinite family of particular solutions, e.g. the N-soliton solution.

(ii) The modified Korteweg–de Vries (MKdV) equation. Similarly, for the MKdV equation

$$E_1 \equiv u_t + \left(u_{xx} - \frac{2}{\alpha^2}u^3\right)_x = 0 \tag{8}$$

$$E_2 \equiv U_t + \left(U_{xx} - \frac{2}{\alpha^2}U^3\right)_x = 0 \tag{9}$$

the auto-BT is given by (Lamb 1974)

$$F_1 \equiv (w+W)_x + 2\alpha\lambda \sinh\frac{w-W}{\alpha} = 0 \tag{10}$$

$$F_2 \equiv (w+W)_t - 8\lambda^2 W_x + 4\lambda W_{xx}\cosh\frac{w-W}{\alpha} - 4\alpha\left(2\lambda^3 - \lambda\frac{W_x^2}{\alpha^2}\right)\sinh\frac{w-W}{\alpha} = 0 \tag{11}$$

where λ is an arbitrary constant and w, W are the potential fields

$$u = w_x \qquad U = W_x \qquad E = \tilde{E}_x \qquad \tilde{E}(w) \equiv w_t + w_{xxx} - 2\frac{w_x^3}{\alpha^2} = 0. \tag{12}$$

Indeed, the elimination of w yields

$$\mathrm{e}^{-(w-W)/\alpha}[(\mathrm{e}^{(w-W)/\alpha}F_1)_t - (\mathrm{e}^{(w-W)/\alpha}F_2)_x] \equiv 2E_2. \tag{13}$$

Remark. The second BT equation (11) can also be written in the symmetric form:

$$(w+W)_t + 2\lambda(W-w)_{xx}\cosh\frac{w-W}{\alpha} + 2\lambda\frac{W_x^2 + w_x^2}{\alpha}\sinh\frac{w-W}{\alpha} = 0 \tag{14}$$

which is invariant under the involution $(w, W, \lambda) \to (W, w, -\lambda)$.

Since the BT results from an elimination process (Wadati *et al* 1975) it is sufficient to obtain both the Lax pair (Lax 1968) and the Darboux transformation (DT) (Darboux 1882) in order to constructively prove integrability.

On the other hand, the study of the singularity structure of the 'general solution' of a PDE (Painlevé analysis (Weiss *et al* 1983)) provides quite important information. If the PDE passes the Painlevé test (a set of necessary conditions for the absence of movable critical singularities in the 'general solution'), it may be integrable. A sufficient proof of integrability is then the explicit construction of a DT and a Lax pair, and Weiss (1983) has succeeded in doing this in an almost algorithmic way with his *singular-manifold method* for many PDEs (KdV, Boussinesq, Sawada–Kotera and others).

However, when one carefully examines the list of PDEs thus processed by Weiss, one notices (Musette and Conte 1991) that this singular-manifold method is successful only when the DT involves *one* singular manifold and is defined as

$$u = U + \mathcal{D}\log\psi \tag{15}$$

in terms of the *singular part operator* $\mathcal{D}$ and the logarithmic derivative of the 'τ-function' ψ which represents the movable singular manifold $\psi = 0$. In particular, for the three PDEs of the Zakharov–Shabat–Ablowitz–Kaup–Newell–Segur scheme (ZS–AKNS; Zakharov and Shabat 1971, Ablowitz *et al* 1974) which have *two* families of movable singularities, namely MKdV, SG and nonlinear Schrödinger (NLS) equations, the BT (Lax pair *and* DT) is not found by Weiss with the partial exception of the NLS where only the Lax pair is found after a skilful computation, however less algorithmic (i.e. the Lax pair could not be found if not known in advance; see details in appendix C).

We present here an extension, already outlined elsewhere (Musette 1994), of the (one-)singular-manifold method of Weiss, which we naturally call the two-singular-manifold method, to PDEs having two families of movable singularities with opposite principal parts such as the MKdV, SG and NLS. The method consists of two stages. The first stage is the derivation of the DT by performing the one-singular-manifold method separately for each of the two families; this DT typically expresses the difference between the two solutions u and U of the PDE as the algebraic sum of the singular parts

$$u = U + \mathcal{D}\log\psi_1 - \mathcal{D}\log\psi_2 \tag{16}$$

defined in terms of $\mathcal{D}$ and the logarithmic derivatives of the 'τ-functions' ψ_1 and ψ_2 of each family.

The second stage consists of obtaining the Lax pair, represented by an equivalent projective Riccati system; since the only information about it is that ψ_1 and ψ_2 must satisfy the same linear system defining the Lax pair, one takes the most general such Riccati system with undetermined coefficients, i.e. in the second-order case of one Riccati component

$$Y_x = R_0 + R_1 Y + R_2 Y^2 \tag{17}$$

$$Y_t = S_0 + S_1 Y + S_2 Y^2 \tag{18}$$

$$Y_{xt} - Y_{tx} \equiv X_0 + X_1 Y + X_2 Y^2 \equiv 0 \tag{19}$$

$$X_0 \equiv R_{0,t} - S_{0,x} + R_1 S_0 - R_0 S_1 = 0 \tag{20}$$

$$X_1 \equiv R_{1,t} - S_{1,x} + 2(R_2 S_0 - R_0 S_2) = 0 \tag{21}$$

$$X_2 \equiv R_{2,t} - S_{2,x} - R_1 S_2 + R_2 S_1 = 0. \tag{22}$$

The components ($Y_1 = Y, \ldots$) of this (generally multi-dimensional) Riccati system are defined in terms of the two functions (ψ_1, ψ_2) representing the two singular manifolds:

$Y_1 = Y = \psi_1/\psi_2, \ldots$. This second stage now consists of generating and solving a small set of determining equations, whose unknowns are the coefficients (R_i, S_i) of the Riccati system, to be found in terms of the PDE solution U which occurs in the DT; these determining equations are generated by inserting the DT (16) into the PDE $E(u, x, t) = 0$ thus resulting in an extension of the Weiss truncation procedure to negative and positive powers.

Finally, the BT is obtained by the elimination of Y between the DT (16) and the Riccati form of the Lax pair, a step which reduces to a simple substitution when the operator $\mathcal{D}$ can be inverted.

Throughout this paper, we use the invariant formalism (Conte 1989) of Painlevé analysis, which is equivalent to the Weiss–Tabor–Carnevale (WTC) one (Weiss *et al* 1983) and presents the advantage of shortening many of the expressions due to its built-in homographic invariance (see appendix A).

Remarks.

(i) The idea of taking account of two τ-functions, instead of just one, dates back to Painlevé (1902), who expressed the general solution of (P2)

$$\text{(P2): } u_{xx} = 2u^3 + xu + \alpha \qquad \alpha = \text{constant} \tag{23}$$

in terms of two entire functions (ψ_1, ψ_2)

$$u = \frac{\mathrm{d}}{\mathrm{d}x}(\log\psi_1 - \log\psi_2). \tag{24}$$

(ii) The advantage of the Riccati form over any other nonlinear pseudopotential to represent the Lax pair has often been emphasized (Martini 1987, Nucci 1989, Musette and Conte 1991).

(iii) There exists no homographic transformation which maps the particular Riccati system $Y = \chi$ of the invariant Painlevé analysis (equations (92) and (93)) onto the above general one ((17) and (18)) for arbitrary values of their coefficients. Indeed, such a transformation conserves the number of cross-derivative conditions, which is one for χ and three for Y. Restricting Y to a homographic transform of χ would miss, for example, the Riccati pseudopotential of the NLS which has two non-zero cross-derivative conditions (142), although this would not miss the SG or MKdV case which have only one non-zero cross-derivative condition (68) or (87) (see equation (88)). It is therefore necessary to go beyond the variable χ. The fundamental reason behind this is that χ contains no more information than *one* singular manifold, as shown by its explicit expression (91). This is reflected in the different possible linearizations of these Riccati systems: the system for Y cannot, in general, be linearized by the transformations $Y = -(1/R_2)\psi_x/\psi$ or $Y = R_0\psi/\psi_x$ which explicitly restrict R_2 or R_0 to never vanish; it can only be linearized by $Y = \psi_1/\psi_2$ into an essentially two-component linear system and is thus well suited to the two-manifold situation. In contrast, the χ system supports both types of linearization but the type $\chi = \psi_1/\psi_2$ should not be understood as involving two different manifolds since in fact $\psi_2 = \psi_{1,x}$.

(iv) The 'squared eigenfunction transformation' (Weiss *et al* 1983)

$$\varphi_x = \psi^{-2} \tag{25}$$

between the singular-manifold function φ of WTC and the solution ψ of the unknown underlying Lax pair, which works so nicely for many one-manifold PDEs, is *not* adapted to the two-manifold PDEs. Indeed, it relies on a property of the Wronskian that ψ and $\psi \int \psi^{-2}\,dx$ are *two independent solutions of the linear ordinary differential equation* (ODE) which defines the first half of the Lax pair. However, such a property explicitly assumes for this ODE the Sturm–Liouville form (94), i.e. it restricts the Riccati equation (17) to $R_1 = 0$ and either R_0 or R_2 to a pure non-zero constant.

(v) A similar extension of the Weiss truncation to negative and positive powers has already been considered (Pickering 1993) but its use of χ instead of Y only allows one to find travelling waves of integrable or non-integrable PDEs.

(vi) *A recent attempt has been made (Estévez et al 1993) to extend the above idea of* Painlevé to PDEs in order to retrieve two Lax pairs. For the Broer–Kaup system (Broer 1975, Kaup 1975, Matveev and Yavor 1979) and the modified Boussinesq equation (Hirota and Satsuma 1977), these authors introduced two manifolds but could not retrieve the correct Lax pair (Fordy and Gibbons 1981) of the modified Boussinesq equation and could only find the restriction of the Broer–Kaup equation (Kaup 1975) to a zero value of the spectral parameter. Moreover, their method was not systematic. Similar ideas have also been used slightly differently (Garagash 1993).

In section 2, we recall the (one-)singular-manifold method of Weiss which, when applied to the KdV equation, i.e. the only one-family equation of the ZS–AKNS class, yields both the DT *and* the Lax pair. This result is needed for it will be used later. Section 3 details the present method, already outlined above. The case of the SG, by far the simplest one, is handled in section 4. The case of the modified KdV, which needs the prerequisite results for the KdV, is treated in section 5.

2. The (one-)singular-manifold method

Consider a PDE, algebraic in u and its partial derivatives

$$E(u, x, t) = 0 \tag{26}$$

which passes the Painlevé test, i.e. which fulfils all the necessary conditions one can build for the absence of movable critical singularities in the 'general solution'. One assumes that it does not admit two families of movable singularities with opposite principal parts. Such a one-family PDE is the KdV equation

$$E \equiv u_t + \left(u_{xx} - \frac{3}{a}u^2\right)_x = 0 \tag{27}$$

admitting the single family $u \sim 2a\chi^{-2}$ with the Fuchs indices ('Painlevé resonances') $-1, 4, 6$

$$u = 2a\chi^{-2} - a\frac{C - 4S}{6} + \frac{a}{6}(C - S)_x\chi + \mathrm{O}(\chi^2). \tag{28}$$

The idea (Weiss *et al* 1983) is that the singular part, i.e. the restriction of this *local* Laurent series to its non-positive powers, contains all the information for a *global* knowledge of the PDE through its DT and its Lax pair. This is quite analogous to the proof of 'intégration parfaite' (achieved) in which Painlevé (1902) expressed the general solution of his first equation (PI)

$$\text{(PI)}: \; u_{xx} = 6u^2 + x \tag{29}$$

in terms of an entire function ψ through a logarithmic derivative

$$u = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} \log \psi. \tag{30}$$

Let $-p$ and $-q$ denote the two positive integers (2 and 5 in our KdV example) equal to the singularity orders of u and E. We consider (Weiss *et al* 1983) the truncated series

$$u_{\mathrm{T}} = \sum_{j=0}^{-p} u_j \chi^{j+p} \qquad E_{\mathrm{T}} = E(u_{\mathrm{T}}) = \sum_{j=0}^{-q} E_j \chi^{j+q} \tag{31}$$

where coefficients $u_j, j \in [0, -p]$ are equal to those of the infinite expansion (28). Coefficients (u_j, E_j) only depend on the homographic invariants (S, C) and the arbitrary coefficients u_i introduced at Fuchs indices i lying in the interval $[0, -p]$. The truncation (31) only exists if the set of overdetermined equations

$$\forall j \in]-p, -q] : \ E_j(S, C, \{u_i, i \text{ Fuchs index } \in [0, -p]\}) = 0 \tag{32}$$

has a non-empty solution.

In our KdV example, one finds

$$E_3 \equiv -2a(C - S)_x = 0 \tag{33}$$

$$12E_5 \equiv -2a(C - S)_t + (C + 2S)E_3 + 4E_{3,xx} = 0 \tag{34}$$

equations equivalent to the single equation

$$C - S - 6\lambda = 0 \qquad \lambda = \text{arbitrary constant} \tag{35}$$

called the *singular-manifold equation* (SME) by Weiss.

The parametric representation of the SME

$$S = -2\frac{U}{a} - 2\lambda \qquad C = -2\frac{U}{a} + 4\lambda \tag{36}$$

provides the Riccati system (92) and (93)

$$(\chi^{-1})_x = -\chi^{-2} + \frac{U}{a} + \lambda \tag{37}$$

$$(\chi^{-1})_t = \left(\left(2\frac{U}{a} - 4\lambda\right)\chi^{-1} - \frac{U_x}{a}\right)_x$$

satisfying the cross-derivative condition (97) $X \equiv -(2/a)\mathrm{KdV}(U) = 0$. This system is linearizable either by $\chi = \psi/\psi_x$ into the second-order linear system ((94) and (95))

$$\psi_{xx} - \left(\frac{U}{a} + \lambda\right)\psi = 0 \tag{39}$$

$$\psi_t + \left(4\lambda - 2\frac{U}{a}\right)\psi_x + \frac{U_x}{a}\psi = 0 \tag{40}$$

or by $\chi = \psi_1/\psi_2$ into the ZS–AKNS matricial Lax pair. The map between two solutions of the KdV results from the equations defining the truncation (31) and the parametric representation (36)

$$u_{\mathrm{T}} = -2a(\log\psi)_{xx} - a\frac{C+2S}{6} \tag{41}$$

$$= -2a(\log\psi)_{xx} + U. \tag{42}$$

In this case, the Weiss truncation yields both the Lax pair ((39) and (40)) and the DT (42).

For a scattering problem of order higher than two, the function ψ in equation (42) is assumed to satisfy a scalar unknown Lax pair (Weiss 1983, Musette and Conte 1991) with the coefficients to be determined.

The one-family truncation equations (32) have been called *Painlevé–Bäcklund equations* (PB) for this ability to generate the DT and the Lax pair and therefore the auto-BT (Lamb 1974) by substitution of the DT (42) $\chi^{-1} = -(w-W)/(2a)$ into the couple (37) and (38) (notation $u_{\mathrm{T}} = u = w_x$, $U = W_x$)

$$a(w+W)_x = -2a^2\lambda + \tfrac{1}{2}(w-W)^2 \tag{43}$$

$$a(w+W)_t = 2W_{xx}(w-W) + 4W_x^2 + 2w_xW_x - 4a\lambda(w-W)_x. \tag{44}$$

The second equation can be made invariant under the exchange of w and W

$$a(w+W)_t = (w-W)(W-w)_{xx} + 2(W_x^2 + w_xW_x + w_x^2). \tag{45}$$

3. The two-singular-manifold method

Consider now the PDE (26) which passes the Painlevé test and admits two families of movable singularities with opposite principal parts such as the MKdV (8)

$$u \sim \pm\alpha\chi^{-1} \qquad \text{indices } -1, 3, 4, \tag{46}$$

or the SG equation (3) put in polynomial form (PSG) ((47) and (48))

$$2(vv_{xt} - v_xv_t) - v^3 + v = 0 \qquad \mathrm{e}^{\mathrm{i}u} = v \tag{47}$$

$$\mathrm{e}^{\pm\mathrm{i}u} = v^{\pm1} \sim -4C\chi^{-2} \qquad \text{indices } -1, 2. \tag{48}$$

The first stage of the method we propose consists of performing the WTC truncation (31) successively for each of the two families. This process is entirely algorithmic up to the generation of the truncation equations $E_j = 0$, $j \in]-p, -q]$ and is summarized in table 1. The resolution of the truncation equations, although non-algorithmic (we will call it almost algorithmic), is quite easy (see appendix B for SG, appendix C for NLS) and leads to the SME. For the MKdV, SG and NLS, which are all invariant by parity, this SME is independent of the choice made among the two opposite families.

Table 1. The results of the Painlevé analysis. The four polynomial PDEs are (27), (8), (47) and (128). The integers (p, q) are defined in (31). The next column lists the Fuchs indices, except -1. Column 'PB' lists the subscripts of the non-identically zero PB equations; in the SG and NLS case, they depend on the arbitrary coefficients introduced at the index 2 (SG) and 0 (NLS).

Name	p	q	Indices	PB eq	SME
KdV	-2	-5	4, 6	2, 5	$C - S - 6\lambda = 0$
MKdV	-1	-4	3, 4	2	$C - S = 0$
SG	-2	-6	2	3, 4, 5, 6	$S + C_{xx}/C - (C_x/C)^2/2 + 2\lambda = 0$
NLS	$(-1, -1)$	$(-3, -3)$	0, 3, 4	2, 2, 3	$C_t + 3CC_x - S_x + 8\lambda C_x = 0$

The second stage consists of exhausting the information contained in the SME in order to obtain first the DT, then the Lax pair. This DT is materialized by finding *another* solution of the PDE, distinct from the truncated variable u_{T}, and this may require the consideration of other intermediate PDEs ('Miura transforms'). Once found, the DT is expressed in terms of the components ($Y_1 = Y, \ldots$) of an undetermined Riccati pseudopotential. This defines a truncation different from that of WTC, extended to negative and positive powers of Y_k. Its resolution is as easy as that of the WTC truncation and presents quite similar features, like identically zero equations in which the subscript is a Fuchs index, or the symmetric of a Fuchs index, with respect to $-q$. Finally, the matricial Lax pair results from the canonical linearization of the projective Riccati system. This is now detailed for two examples.

4. The sine–Gordon case

The SG equation (3) is invariant by parity on u and by exchange of ∂_x and ∂_t so any result must have this invariance.

Let us denote by $(\chi_i, \psi_i, s_i, C_i)$, $i = 1, 2$ the functions (χ, ψ, S, C) pertaining to each of the two families of movable singularities $v \sim -4C_1\chi_1^{-2}$ and $v^{-1} \sim -4C_2\chi_2^{-2}$. The PB equations have the following general solution (Weiss 1984, Conte 1989) (see appendix B). For the first family

$$C_1 = -\frac{V}{4\lambda} = -\frac{\mathrm{e}^{\mathrm{i}U}}{4\lambda} \qquad \mathrm{PSG}(V) = 0 \qquad \mathrm{SG}(U) = 0 \tag{49}$$

$$s_1 = -\frac{V_{xx}}{V} + \frac{V_x^2}{2V^2} - 2\lambda = -\mathrm{i}U_{xx} + \tfrac{1}{2}U_x^2 - 2\lambda \tag{50}$$

$$\mathrm{e}^{\mathrm{i}u} = v = -4(\log\psi_1)_{xt} + V. \tag{51}$$

For the second family $\mathrm{e}^{-\mathrm{i}u} = w \sim -4C_2\chi_2^{-2}$

$$C_2 = -\frac{W}{4\lambda} = -\frac{\mathrm{e}^{-\mathrm{i}U}}{4\lambda} \qquad \mathrm{PSG}(W) = 0 \qquad \mathrm{SG}(U) = 0 \tag{52}$$

$$s_2 = -\frac{W_{xx}}{W} + \frac{W_x^2}{2W^2} - 2\lambda = \mathrm{i}U_{xx} + \tfrac{1}{2}U_x^2 - 2\lambda \tag{53}$$

$$\mathrm{e}^{-\mathrm{i}u} = w = -4(\log\psi_2)_{xt} + W. \tag{54}$$

If one considers only one of these two equivalent singular manifolds, the cross-derivative condition (97) is evaluated to $\mathrm{e}^{-\mathrm{i}U}(\mathrm{e}^{\mathrm{i}U}\mathrm{SG}(U))_x$ and not simply to $\mathrm{SG}(U)$ which proves that the one-singular-manifold method does not provide the correct Lax pair. More precisely, from this Lax pair, the N-soliton solution could not be generated by the Crum procedure (Crum 1955).

The algebraic sum of the two opposite singular parts is best computed on $\sin u$, which has the same parity as u, from the two equations (51) and (54)

$$\sin u = 2\mathrm{i}((\log\psi_1)_{xt} - (\log\psi_2)_{xt}) + \sin U \tag{55}$$

an equation which, from the definition of the SG PDE, is equivalent to

$$u_{xt} = 2\mathrm{i}((\log\psi_1)_{xt} - (\log\psi_2)_{xt}) + U_{xt}. \tag{56}$$

Integrating twice, we finally obtain the DT of the SG

$$u = 2\mathrm{i}(\log\psi_1 - \log\psi_2) + U \tag{57}$$

defined in terms of *both* families. In terms of $Y = \psi_1/\psi_2$, the 'truncation' is

$$u = 2\mathrm{i}\log Y + U \qquad \mathrm{e}^{\mathrm{i}u} = Y^{-2}\mathrm{e}^{\mathrm{i}U} \qquad Y = \frac{\psi_1}{\psi_2} \qquad \mathrm{SG}(u) = 0 \qquad \mathrm{SG}(U) = 0 \tag{58}$$

and one must identify

$$\mathrm{SG}(u) - \mathrm{SG}(U) \equiv \sum_{j=0}^{4} E_j Y^{j-2} \tag{59}$$

to the null polynomial in Y. The set of five PB equations is invariant under the involution

$$(Y, R_k, S_k, U) \to (Y^{-1}, -R_{2-k}, -S_{2-k}, -U) \qquad k = 0, 1, 2 \tag{60}$$

$$E_0 \equiv 2R_0S_0 - \frac{V}{2} = 0 \tag{61}$$

$$E_1 \equiv 2R_1S_0 - 2S_{0,x} = 0 \tag{62}$$

$$E_2 \equiv 2R_2S_0 - 2R_0S_2 - \frac{1}{2V} + \frac{V}{2} - 2S_{1,x} = 0 \tag{63}$$

$$E_3 \equiv -2R_1S_2 - 2S_{2,x} = 0 \tag{64}$$

$$E_4 \equiv -2R_2S_2 + \frac{1}{2V} = 0 \tag{65}$$

and the absence of movable logarithms at index 2 is reflected in E_2 being a differential consequence of (E_0, E_1) or (E_4, E_3) as well, *modulo* the conditions (20)–(22). One obtains $(S_0S_2)_x = 0$ from E_1, E_3, then $(R_0R_2)_t = 0$ from the two cross-derivative conditions (20) and (22), and $R_0R_2S_0S_2 = -1/16$ from E_0, E_4. This introduces an arbitrary constant $\mu = S_0S_2$ and the third cross-derivative condition $X_1 = 0$ (21) expresses the fact that $-S_0^2/(\mu V)$ is a solution of (47), i.e. equal to either V or $1/V$ (the value ± 1 would make X_1 identically zero and, hence, it would not provide a pseudopotential). For $-S_0^2/(\mu V) = V$ the unique solution is the Riccati pseudopotential

$$Y_x = \lambda(1 - Y^2) + \frac{V_x}{V}Y = \lambda(1 - Y^2) + \mathrm{i}U_xY \qquad \mu = -16\lambda^2 \tag{66}$$

$$4\lambda Y_t = V - \frac{Y^2}{V} = (1 - Y^2)\cos U + \mathrm{i}(1 + Y^2)\sin U \tag{67}$$

$$\frac{Y_{xt} - Y_{tx}}{Y} = \mathrm{i}\,\mathrm{SG}(U) \tag{68}$$

while the choice $-S_0^2/(\mu V) = 1/V$ provides the pseudopotential obtained by exchanging ∂_x and ∂_t. In contrast to equation (92) with S given by equation (50), equation (66) is now linear in λ and U and able to (Salle 1982, Matveev and Salle 1991) generate the N-soliton solution, as required for a 'good' Lax pair.

As to the BT (5), it is readily obtained by substituting $Y = \exp(-(\mathrm{i}/2)(u - U))$ into the two equations (66) and (67).

Remark. The physical equation SG (3) does not admit a one-family truncation but only a two-family one while the non-physical but polynomial equation PSG (47) admits both.

5. The modified KdV case

Equation (8) has two families $u \sim \pm\alpha\chi^{-1}$ which we simply denote $u \sim \alpha\chi^{-1}$ since α is only defined by its square. The truncated expansion of any one of the two families reduces to

$$u_{\mathrm{T}} = \alpha\chi^{-1}. \tag{69}$$

The SME $S - C = 0$ is parametrized as

$$S = -2(v/a) \qquad C = -2(v/a) \qquad \mathrm{KdV}(v) = 0 \tag{70}$$

and equations (94) and (95) fail to define a Lax pair for the MKdV because the cross-derivative condition (97) evaluates to $(\partial_x + (2/\alpha)\mathrm{MKdV}(u_{\mathrm{T}}))\mathrm{MKdV}(u_{\mathrm{T}})$ and not simply to $\mathrm{MKdV}(u_{\mathrm{T}})$ as it should. The precise transformation between u and v (Miura transformation) is obtained by eliminating χ between (69) and (92)

$$\left(\frac{u_{\mathrm{T}}}{\alpha}\right)_x + \left(\frac{u_{\mathrm{T}}}{\alpha}\right)^2 = \frac{v}{a}. \tag{71}$$

In fact there are two such Miura transformations, one for each sign of α, i.e. one for each family. Equation (69) involves only one solution of the MKdV, not two as in the SG case (51), and thus it also fails to provide a DT.

Let us first obtain the DT for the MKdV from that of the KdV and show that it involves two singular manifolds. The DT for the KdV has been obtained in section 2, equation (42). The two Miura transformations (71) and the parametrization (70) imply

$$-\frac{s_1}{2} = \left(\frac{u_{\mathrm{T}}}{\alpha}\right)^2 - \left(\frac{u_{\mathrm{T}}}{\alpha}\right)_x = -2(\log\psi_1)_{xx} + \left(\frac{U}{\alpha}\right)^2 - \left(\frac{U}{\alpha}\right)_x \tag{72}$$

$$-\frac{s_2}{2} = \left(\frac{u_{\mathrm{T}}}{\alpha}\right)^2 + \left(\frac{u_{\mathrm{T}}}{\alpha}\right)_x = -2(\log\psi_2)_{xx} + \left(\frac{U}{\alpha}\right)^2 + \left(\frac{U}{\alpha}\right)_x \tag{73}$$

and taking the algebraic sum of the singular parts eliminates the nonlinear terms

$$u_{\mathrm{T},x} = \alpha((\log\psi_1)_{xx} - (\log\psi_2)_{xx}) + U_x \tag{74}$$

a relation which, after one integration, yields the DT for the MKdV

$$u = \alpha((\log\psi_1)_x - (\log\psi_2)_x) + U \qquad u = u_{\mathrm{T}}. \tag{75}$$

With this DT, the Lax pair is obtained as explained in section 3. Setting $Y = \psi_1/\psi_2$ and taking account of (17) and (18), every derivative of Y evaluates to a polynomial in Y. Consequently, the DT (75) identifies to

$$u_{\mathrm{T}} = \alpha(R_0 Y^{-1} + R_1 + R_2 Y) + U \tag{76}$$

and one must identify

$$E(u_{\mathrm{T}}) - E(U) \equiv \sum_{j=0}^{8} E_j Y^{j-4} \tag{77}$$

to the null polynomial in Y. Equations $E_0 = 0$ and $E_8 = 0$ are, by construction, identically zero since the exact principal part $\alpha Y_x/Y$ was inserted. The combinations $E_j \pm E_{8-j}$ of the PB equations have parity $\pm(-1)^{j+1}$ under the involution

$$(Y, R_k, S_k, U) \to (Y^{-1}, -R_{2-k}, -S_{2-k}, -U) \qquad k = 0, 1, 2 \tag{78}$$

and the Fuchs indices 3 and 4 make not only the equations $E_3 = 0$, $E_4 = 0$ identically zero, as in the WTC truncation, but also $E_{8-j} = 0$, $j = 3, 4$, i.e. also $j = 5$. Finally, among the nine PB equations $E_j = 0$, only four ($j = 1, 2, 6, 7$) are not identically zero:

$$E_1 \equiv 6\frac{R_{0,x}}{R_0} + 6R_1 + 12\frac{U}{\alpha} = 0 \tag{79}$$

$$E_2 \equiv \left(2R_1 - 12\frac{U}{\alpha}\right)\frac{R_{0,x}}{R_0} - 3\frac{R_{0,x}^2}{R_0^2} - 4\frac{R_{0,xx}}{R_0} - 2R_{1,x} - 6\frac{U_x}{\alpha} + 11R_1^2 + 4R_0R_2 - \frac{S_0}{R_0}$$
$$+ 24R_1\frac{U}{\alpha} + 6\frac{U^2}{\alpha^2} = 0 \tag{80}$$

$$E_6 \equiv \left(2R_1 - 12\frac{U}{\alpha}\right)\frac{R_{2,x}}{R_2} + 3\frac{R_{2,x}^2}{R_2^2} + 4\frac{R_{2,xx}}{R_2} - 2R_{1,x} - 6\frac{U_x}{\alpha} - 11R_1^2 - 4R_0R_2 + \frac{S_2}{R_2}$$
$$- 24R_1\frac{U}{\alpha} - 6\frac{U^2}{\alpha^2} = 0 \tag{81}$$

$$E_7 \equiv 6\frac{R_{2,x}}{R_2} - 6R_1 - 12\frac{U}{\alpha} = 0 \tag{82}$$

and they are immediately solved as

$$\frac{R_{0,x}}{R_0} = -R_1 - 2\frac{U}{\alpha} \qquad \frac{S_0}{2R_0} = 2R_0R_2 + \left(R_1 + \frac{U}{\alpha}\right)^2 + \left(R_1 + \frac{U}{\alpha}\right)_x \tag{83}$$

$$\frac{R_{2,x}}{R_2} = \quad R_1 + 2\frac{U}{\alpha} \qquad \frac{S_2}{2R_2} = 2R_0R_2 + \left(R_1 + \frac{U}{\alpha}\right)^2 - \left(R_1 + \frac{U}{\alpha}\right)_x . \tag{84}$$

One thus obtains $(R_0R_2)_x = 0$ from these equations and $(R_0R_2)_t = 0$ from the two cross-derivative conditions $X_0 = 0$ and $X_2 = 0$, equations (20) and (22). This introduces an arbitrary constant which we denote $-\lambda^2 = R_0R_2$ and the third cross-derivative condition $X_1 = 0$ (21) expresses the condition that $U + \alpha R_1$ must satisfy the MKdV equation, i.e. be equal to $\varepsilon U, \varepsilon = \pm 1$ (the choice $U + \alpha R_1 = 0$ does not provide a pseudopotential). Changing Y to $\lambda Y/R_0$, we finally obtain a unique solution as a parametric representation of the six unknowns R_i, S_i in which R_i is linear in U and the spectral parameter λ

$$Y_x = \lambda(1 - Y^2) - 2\frac{\varepsilon U}{\alpha}Y \tag{85}$$

$$Y_t = \lambda\left(2\frac{U^2}{\alpha^2} - 4\lambda^2\right)(1 - Y^2) - 2\lambda\frac{\varepsilon U_x}{\alpha}(1 + Y^2) + \varepsilon\left(8\lambda^2\frac{U}{\alpha} - 4\frac{U^3}{\alpha^3} + 2\frac{U_{xx}}{\alpha}\right)Y$$
$$= \left(-4\lambda\frac{\varepsilon U}{\alpha} + \left(2\frac{U^2}{\alpha^2} - 4\lambda^2 + 2\frac{\varepsilon U_x}{\alpha}\right)Y\right)_x \tag{86}$$

$$\alpha X_1 \equiv -2\varepsilon \mathrm{MKdV}(U) \qquad X_0 \equiv 0 \qquad X_2 \equiv 0. \tag{87}$$

As to the BT (10) and (11), it is obtained by substituting $Y = \exp((w - W)/\alpha)$ into the two equations (85) and (86) with the choice $\varepsilon = 1$.

Remarks.

(i) The Lax pair found by Weiss (1983) is, in fact, that of the KdV, *not* that of the MKdV, even after introduction of the missing spectral parameter λ by exploiting the Galilean invariance of the KdV. Indeed, even after the replacement of the solution of the KdV in terms of that of the MKdV in (39) and (40) using the Miura transformation (71), the cross-derivative condition (97) evaluates to $(\partial_x + (2/\alpha)\mathrm{MKdV}(U))\mathrm{MKdV}(U)$.

(ii) Under the *ad hoc* homographic transformation (Musette 1991)

$$\chi^{-1} = \frac{U}{\alpha} + \lambda Y \qquad \mu = \lambda^2 \tag{88}$$

the Riccati pseudopotential of KdV (37) and (38) (with (U, λ) denoted (V, μ) to avoid conflicting notation) is transformed into that of MKdV (85) and (86), after the replacement (71) $V/a = (U/\alpha)^2 + U_{xx}/\alpha$. The quadratic term U^2 of (37) has now disappeared from (85) and the cross-derivative condition $(Y_{xt} - Y_{tx})/Y$ is $\mathrm{MKdV}(U)$.

One important feature to notice is the different linearization of these two Riccati pseudopotentials. That of KdV (37) and (38) can be linearized either by $\chi = \psi/\psi_x$ or by $\chi = \psi_1/\psi_2$ into, respectively, the scalar form of the Lax pair or the ZS-AKNS matricial form, equivalent under $\psi_1 = \psi$, $\psi_2 = \psi_x$. In contrast, that of the MKdV *cannot* be linearized by $Y = \psi/(\lambda\psi_x)$, a transformation which would explicitly restrict λ to be non-zero.

(iii) The elimination of the linear term u_{T} between (72) and (73) leads to

$$U_{\mathrm{T}}^2 = -\alpha^2(\log(\psi_1\psi_2))_{xx} + U^2 \tag{89}$$

consistent with the bilinear representation of Hirota (1972).

6. Conclusion

Forthcoming papers will handle seemingly more complicated equations, such as the NLS, the Broer–Kaup system or the 'second modified' KdV equation (Fokas 1980, Nakamura and Hirota 1980, Calogero and Degasperis 1981). The present method could also be applied to higher-dimensional equations, such as the $(2+1)$-dimensional generalization of the SG equation (Konopelchenko and Rogers 1991).

Appendix A. Invariant Painlevé analysis

This consists of a built-in resummation of the Laurent series in order to generate the shortest possible coefficients without losing any information. Given a movable singular manifold (φ denotes a function, φ_0 an arbitrary constant)

$$\varphi - \varphi_0 = 0 \tag{90}$$

the expansion variable χ, which must vanish as $\varphi - \varphi_0$, is chosen to be (Conte 1989)

$$\chi = \frac{\psi}{\psi_x} = \left(\frac{\varphi_x}{\varphi - \varphi_0} - \frac{\varphi_{xx}}{2\varphi_x}\right)^{-1} \qquad \psi = (\varphi - \varphi_0)\varphi_x^{-1/2}. \tag{91}$$

The movable singular manifold can thus equivalently be represented as (90) or $\chi = 0$ or $\psi = 0$. The variable χ satisfies the Riccati equations

$$\chi_x = 1 + \tfrac{1}{2} S \chi^2 \tag{92}$$

$$\chi_t = -C + C_x \chi - \tfrac{1}{2}(CS + C_{xx})\chi^2 \tag{93}$$

and the variable ψ the linear equations

$$\psi_{xx} + \tfrac{1}{2} S \psi = 0 \tag{94}$$

$$\psi_t + C\psi_x - \tfrac{1}{2} C_x \psi = 0 \tag{95}$$

the coefficients of which only depend on two functions of the derivatives of φ

$$S = \{\varphi; x\} = \left(\frac{\varphi_{xx}}{\varphi_x}\right)_x - \frac{1}{2}\left(\frac{\varphi_{xx}}{\varphi_x}\right)^2 \qquad C = -\frac{\varphi_t}{\varphi_x} \tag{96}$$

linked by the cross-derivative condition

$$X \equiv S_t + C_{xxx} + 2C_x S + C S_x = 0. \tag{97}$$

The coefficients (u_j, E_j) of the Laurent series in χ only depend on (S, C) and their derivatives (plus the arbitrary coefficients u_i introduced at the Fuchs indices i) and are therefore invariant under the group of homographic transformations $\varphi \to (a\varphi + b)/(c\varphi + d)$, $((a, b, c, d) =$ arbitrary complex constants).

The function ψ has movable zeros as the only movable singularities and, consequently, the expansion variable χ, which depends on one arbitrary function, has movable poles and movable zeros.

Appendix B. One-family truncation of the ZS equation

The SG equation belongs to a group of equations isolated by Zhiber and Shabat (1979)

$$u_{xt} + \alpha e^u + a_1 e^{-u} + a_0 e^{-2u} = 0 \qquad (\alpha, a_1, a_0) = \text{constants} \qquad \alpha \neq 0 \tag{98}$$

which passes the Painlevé test only in the three cases $(a_1 = a_0 = 0)$ (Liouville), $(a_1 \neq 0, a_0 = 0)$ (SG or sinh–Gordon, with identical singularities), $(a_1 = 0, a_0 \neq 0)$ (Tzitzéica or Dodd–Bullough–Mikhailov (DBM) (Tzitzéica 1910a,b, Dodd and Bullough 1976, Mikhailov 1981)). We here discard the case of the linearizable Liouville equation.

The only dependent variables equal to a power of e^u which, at the same time, make the ZS equation polynomial and the leading power p a negative integer are e^u and e^{-u}. In the SG case they are equivalent and, in the DBM case, the PDE for e^{-u} $(p = -1, q = -4$, indices $(-1, 2)$, $E_3 = 0, E_4 = 0)$ has no solution (Weiss 1986). Let us therefore perform the one-family truncation of e^u in the case $(a_1, a_0) \neq (0, 0)$. The polynomial PDE satisfied by the variable $e^u = v$

$$E(v) \equiv v v_{xt} - v_x v_t + \alpha v^3 + a_1 v + a_0 = 0 \tag{99}$$

has the one-family truncation ($p=-2, q=-6$, indices $-1, 2$)

$$\alpha v_{\mathrm{T}} = 2C\chi^{-2} - 2C_x\chi^{-1} + \text{arbitrary coefficient} \tag{100}$$

and we define the arbitrary coefficient v_2 arising at the Fuchs index 2 with a convenient translation aimed at making $v_{\mathrm{T}} - v_2$ invariant under the permutation $\mathcal{P}(\partial_x, \partial_t) \to (\partial_t, \partial_x)$ which leaves the ZS equation invariant

$$\alpha v_{\mathrm{T}} = 2C\chi^{-2} - 2C_x\chi^{-1} + \alpha v_2 + SC + C_{xx} - \tfrac{1}{2}(\log C)_{xt}. \tag{101}$$

The four PB equations $E_j = 0$, $j = 3, 4, 5, 6$ depend on (S, C, v_2) and the first three are linear in $(v_{2,x}, v_{2,t}, v_{2,xt})$ with a Jacobian equal to $J = (\log C)_{xt}$. This makes the resolution extremely easy.

The case $J = 0$ implies S and C constant (after some algebra) and leads to the solutions (Conte and Musette 1992)

$$\forall(a_0, a_1):\quad \alpha v = 2c\frac{k^2}{4}\left(\tanh^2\frac{k}{2}(x - ct - x_1) - 1\right) + \alpha c_0$$
$$\alpha c_0^3 + a_1 c_0 + a_0 = 0 \qquad 4ck^2 = \alpha c_0 - a_1 c_0^{-1} - 2a_0 c_0^{-2} \tag{102}$$

$$\text{(DBM)}:\quad \alpha v = 2c\frac{k^2}{4}\left(\tanh^2\frac{k}{2}(x - ct - x_1) - \frac{1}{6}\right)$$
$$-2c\wp\left(x + ct - x_2, \frac{k^4}{12} + \frac{\alpha a_1}{c^2}, -\frac{k^6}{216} + \frac{\alpha a_1}{6c^2} - \frac{\alpha^2 a_0}{4c^3}\right) \tag{103}$$

where $\wp$ is the Weierstrass elliptic function. These solutions depend on two and four arbitrary constants, respectively. Therefore, the DBM equation (103) does admit a truncation with a non-empty solution, provided one considers e^u and not e^{-u}.

Remark. The solution (103) is a degeneracy of the exact solution to the DBM equation

$$\alpha v = 2c\wp\left(x - ct - x_1, g_2, A + \frac{\alpha^2 a_0}{8c^3}\right) - 2c\wp\left(x + ct - x_2, g_2, A - \frac{\alpha^2 a_0}{8c^3}\right) \tag{104}$$

depending on five arbitrary constants (x_1, x_2, c, g_2, A) and representing the superposition of two travelling waves of opposite velocities. This solution is, to our knowledge, new.

In the generic case $J \neq 0$, the four PB equations are algebraically equivalent to

$$\alpha v_{2,x} = K_1 \alpha v_2 + K_2 \tag{105}$$

$$\alpha v_{2,t} = K_3 \alpha v_2 + K_4 \tag{106}$$

$$\alpha v_{2,xt} = -\alpha a_1 - 3(\alpha v_2)^2 + K_5 \alpha v_2 + K_6 \tag{107}$$

$$\alpha^2 E_6 \equiv \alpha^2 a_0 - 2(\alpha v_2 - A_1)(\alpha v_2 - A_2)(\alpha v_2 - A_3) = 0 \tag{108}$$

where K_n and A_n denote short expressions of (S, C) independent of (α, a_1, a_0). Due to the convenient translation, the A_n are invariant under $\mathcal{P}$ ($A_1 = -A_2 = J/2$ is odd, A_3 is even). The Schwarz cross-derivative conditions provide two other relations

$$\alpha((v_{2,x})_t - v_{2,xt}) \equiv \alpha a_1 + 3(\alpha v_2 - A_2)(\alpha v_2 - A_4) = 0 \tag{109}$$

$$\alpha((v_{2,t})_x - v_{2,xt}) \equiv \alpha a_1 + 3(\alpha v_2 - A_1)(\alpha v_2 - A_5) = 0 \tag{110}$$

where A_4 and A_5 are also independent of (α, a_1, a_0) and exchanged under $\mathcal{P}$. The useful expressions are

$$2A_1 = J = -2A_2 \tag{111}$$

$$2A_3 = J - 2(SC + C_{xx}) - 2J^{-1}(J_x C_x + C_x S_t) - J^{-2}(J_x^2 C + 2J_x C S_t + C S_t^2) \tag{112}$$

$$6A_4 = 3J + 2J^{-1}(S_{tt} - C^{-1}C_t S_t - C_x S_t) - 2J^{-2}(J_x C S_t + C S_t^2) \tag{113}$$

$$6A_5 = -J - 4(SC + C_{xx}) - 2J^{-1}(J_{xt} + S_{tt} + 2J_x C_x + 2S_t C_x)$$
$$+ 2J^{-2}(J_t J_x - C J_x^2 + J_t S_t - 2C J_x S_t - C S_t^2). \tag{114}$$

In the DBM case, this one-manifold truncation is equivalent to

$$\alpha v_2 = A_4 = A_5 = \tfrac{1}{2}(A_4 + A_5) \tag{115}$$

$$A_{4,x} - K_1 A_4 - K_2 = 0 \tag{116}$$

$$A_{5,t} - K_3 A_5 - K_4 = 0 \tag{117}$$

$$A_4 - A_5 = 0 \tag{118}$$

$$\alpha^2 a_0 - (A_5 - A_1)(A_4 - A_2)(A_4 + A_5 - 2A_3) = 0 \tag{119}$$

equations currently under examination in order to possibly obtain the BT of the DBM equation (Tzitzéica 1910a,b, Gaffet 1988, Andreev 1989, Safin and Sharipov 1993), i.e. its Lax pair and its DT.

In the SG case, the one-manifold truncation equations are equivalent to

$$\alpha v_2 = A_3 \tag{120}$$

$$A_{3,x} - K_1 A_3 - K_2 = 0 \tag{121}$$

$$A_{3,t} - K_3 A_3 - K_4 = 0 \tag{122}$$

$$\alpha a_1 + 3(A_3 - A_2)(A_3 - A_4) = 0 \tag{123}$$

$$\alpha a_1 + 3(A_3 - A_1)(A_3 - A_5) = 0 \tag{124}$$

and their general solution (Weiss 1984, Conte 1989) depends on one arbitrary constant λ

$$S = \frac{V_x^2}{2V^2} - \frac{V_{xx}}{V} - 2\lambda \qquad C = \frac{\alpha V}{2\lambda} \qquad v_2 = \frac{V}{2} - \frac{a_1}{2\alpha V} \tag{125}$$

$$X \equiv -\frac{1}{V}\left(\frac{E(V)}{V}\right)_x = 0. \tag{126}$$

Appendix C. One-family truncation of the NLS equation

Instead of the NLS equation

$$E \equiv \mathrm{i}u_t + u_{xx} + a|u|^2 u = 0 \qquad a \in \mathcal{R} \qquad a \neq 0 \qquad u \in \mathcal{C} \tag{127}$$

let us process the system (Zakharov and Shabat 1971, Ablowitz *et al* 1974)

$$\begin{aligned} &\mathrm{i}u_t + u_{xx} + au^2 v = 0 \\ -&\mathrm{i}v_t + v_{xx} + auv^2 = 0 \end{aligned} \tag{128}$$

in which the reduction $v = \bar{u}$ is the NLS.

The one-family truncation

$$u_\mathrm{T} = u_0 \chi^{-1} + u_1 \tag{129}$$

$$v_\mathrm{T} = v_0 \chi^{-1} + v_1 \tag{130}$$

has been performed by Weiss (1985) and his results expressed in the invariant formalism (appendix A) are

$$u_\mathrm{T} = A_0 \mathrm{e}^{\mathrm{i}w} \left(\chi^{-1} - \mathrm{i}w_x + \tfrac{1}{2}\mathrm{i}C\right) \qquad v_\mathrm{T} = B_0 \mathrm{e}^{-\mathrm{i}w} \left(\chi^{-1} + \mathrm{i}w_x - \tfrac{1}{2}\mathrm{i}C\right) \tag{131}$$

$$F_2 \equiv S - \tfrac{1}{2}C^2 - w_t + 2Cw_x - 3w_x^2 = 0 \qquad A_0 B_0 = -2/a \tag{132}$$

$$G_2 \equiv \mathrm{i}(C - w_x)_x = 0 \tag{133}$$

$$F_3 \equiv w_{xt} + 3w_x w_{xx} - \tfrac{1}{2}Cw_{xx} - \tfrac{1}{2}S_x - \tfrac{1}{2}C_t - w_x C_x = 0 \tag{134}$$

where w is the arbitrary function associated with the index 0 and $F_j, G_j, j = 2, 3$ the truncation coefficients (31) of the half sum and the half difference of equations (128). Their resolution is achieved by the elimination of S which provides the relation

$$-F_{2,x} + \mathrm{i}CG_2 - 2F_3 \equiv (C - w_x)_t = 0 \tag{135}$$

and introduces an arbitrary constant $2\lambda = w_x - C$.

The elimination of w yields the SME for (S, C)

$$w_x = C + 2\lambda \qquad w_t = S - \tfrac{3}{2}C^2 - 8\lambda C - 12\lambda^2 \tag{136}$$

$$C_t + 3CC_x - S_x + 8\lambda C_x = 0 \tag{137}$$

while the elimination of (S, C) yields a PDE for w

$$S = \tfrac{3}{2}w_x^2 + 2\lambda w_x + w_t + 2\lambda^2 \qquad C = w_x - 2\lambda \tag{138}$$

$$w_{xxxx} + 6w_x^2 w_{xx} + 4w_x w_{xt} + 2w_t w_{xx} + w_{tt} = 0 \tag{139}$$

which is nothing other than the Broer–Kaup equation (Broer 1975, Kaup 1975). Thus, as for the MKdV, a second solution to (128) is still missing at this point in order to define a DT, and the application of the present method to the Broer–Kaup equation (Conte *et al* 1994) may clarify this point.

As to the Lax pair, it was found by Weiss (1985) without the help of a DT, in a way which looks rather complicated at first glance but which results in fact from the elimination of χ between the three equations (131) and (92): the antisymmetric combination $u_{\mathrm{T}}/(A_0\varphi_0) - (v_{\mathrm{T}}/B_0)\varphi_0$, where $\varphi_0 = \mathrm{e}^{\mathrm{i}w}$, directly yields the x part

$$\varphi_{0,x} + 2\mathrm{i}\lambda\varphi_0 + \frac{u}{A_0} - \frac{v}{B_0}\varphi_0^2 = 0 \qquad u = u_{\mathrm{T}} \qquad v = v_{\mathrm{T}} \tag{140}$$

and the symmetric combination $u_{\mathrm{T}}/(A_0\varphi_0) + (v_{\mathrm{T}}/B_0)\varphi_0$ defines a value for χ which, after replacement in (92), yields the t part

$$-\mathrm{i}\varphi_{0,t} + \left(\frac{2uv}{A_0B_0} + 4\lambda^2\right)\varphi_0 + \frac{u_x - 2\mathrm{i}\lambda u}{A_0} + \frac{v_x + 2\mathrm{i}\lambda v}{B_0}\varphi_0^2 = 0 \tag{141}$$

with the cross-derivative condition

$$(\log\varphi_0)_{tx} - (\log\varphi_0)_{xt} = -\frac{\mathrm{i}u_t + u_{xx} + au^2v}{A_0\varphi_0} + \frac{-\mathrm{i}v_t + v_{xx} + auv^2}{B_0}\varphi_0. \tag{142}$$

Therefore, the Riccati pseudopotential φ_0 is simply the exponential of a solution of the Broer–Kaup equation, whose link with NLS is well known (Hirota and Satsuma 1977, Matveev and Yavor 1979).

The only criticism one can make of this nice derivation of the Lax pair by Weiss is the elimination of χ, which is obviously *ad hoc* for the NLS.

Truncation (131) does not apparently involve another solution to (128), distinct from $(u_{\mathrm{T}}, v_{\mathrm{T}})$, and this is reflected by the dependence of equations (140) and (141) on $(u_{\mathrm{T}}, v_{\mathrm{T}})$ and not on this other missing solution. Therefore other developments are necessary in order to find the DT by singularity analysis only.

Remark. The Riccati pseudopotential equations (140) and (141) are invariant under the involution

$$\left(\varphi_0, \frac{u}{A_0}, \frac{v}{B_0}, \lambda\right) \to \left(\frac{1}{\varphi_0}, \frac{v}{B_0}, \frac{u}{A_0}, -\lambda\right) \tag{143}$$

accordingly, if one defines

$$\begin{aligned} u &= A_0Y\left(-\partial_x \log Y + \frac{V}{B_0}Y - 2\mathrm{i}\lambda\right) \\ v &= \frac{B_0}{Y}\left(\partial_x \log Y - \frac{U}{A_0Y} + 2\mathrm{i}\lambda\right) \end{aligned} \tag{144}$$

with (U, V) another solution of (128) and Y a solution of the Riccati equations (17) and (18), these expressions (144) generate an extended truncation quite similar to that of the SG or the MKdV. This truncation is found to admit as a *unique* solution the above Riccati pseudopotential where (φ_0, u, v) is replaced by (Y, U, V). Therefore, equations (144) are as able to generate a Lax pair as equations (58) and (76) which, for the SG and the MKdV, define the DT. Unfortunately, equations (144) fail to define a DT because equations (144) and (140) altogether imply $(u, v) = (U, V)$, i.e. these solutions are not two different ones. However, from the point of view of singularity structure, truncation (144) is quite nice since it is expressed in terms of only two τ-functions, instead of the usual four (Chen 1974, Lamb 1974, Konno and Wadati 1975, Levi *et al* 1984, Neugebauer and Meinel 1984).

To conclude this NLS case, finding the DT and the Lax pair in a purely algorithmic way based only on the singularity structure, as for the SG and the MKdV, is still an open question.

Remark. The transformation $\varphi_0 = \psi_1/\psi_2$ linearizing the Riccati pseudopotential into the ZS–AKNS matricial Lax pair should be related to the DT for the Broer–Kaup equation.

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